Note on the Parallelogram Law

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Abstract

We note an identity in Hilbert Spaces that generalizes the usual parallelogram law. In geometry, this result is called Stewart's Theorem.

1 Result

Let H be a Hilbert space over the real or complex scalars. Then for any x and y in H we have the identity

$$2\|x\|^2 + 2\|y\|^2 = \|x+y\|^2 + \|x-y\|^2.$$

In geometric terms, the sum of the squared lengths of the diagonals of a parallelogram equals the sum of squared lengths of all 4 sides. More generally, let p and q be positive real numbers such that $p+q=1$. Then we have

$$p\|x\|^2 + q\|y\|^2 = \|\bar{x}\|^2 + pq\|x-y\|^2, \ \bar{x} = px + qy. \tag{1.1}$$

For the proof, add and subtract $\bar{x}$ from x and from y and use the fact that $p(x-\bar{x}) + q(y-\bar{x}) = 0$. Then use the fact that $\|x-\bar{x}\| = q\|x-y\|$ and $\|y-\bar{x}\| = p\|x-y\|$.

The classical parallelogram law is the special case $p = q = 1/2$.

The parallelogram law is equivalent to a theorem about medians of triangles due to Apollonius of Perga. Matthew Stewart (c.1717-1785) extended the theorem of Apollonius to more general cevians, and our identity (1.1) is equivalent to his theorem. See, for example, page 6 of [1].

As an example, consider a body of mass m_1 moving at constant velocity v_1 and another body of mass m_2 moving at constant velocity v_2. Then (1.1) yields

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}(m_1+m_2)v_{cm}^2 + \frac{1}{2}\mu(v_1-v_2)^2,$$

where

$$\mu = \frac{m_1m_2}{m_1+m_2}.$$

This provides a decomposition of the total kinetic energy of the two bodies. The first term on the right represents the kinetic energy due to motion of the center of mass, and the second term represents the total kinetic energy in an inertial frame of reference where the center of mass is at rest.

References

[1] H.S.M Coxeter and S.L. Greitzer , *Geometry Revisited*, New Mathematical Library 19, Random House, New York, 1967.