The Fermat-Roberval Inequalities

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Abstract

The Fermat-Roberval inequalities provide an attractive alternative to power sum formulas in the derivation of the standard quadratures of introductory calculus.

1 Introduction

Quadrature problems, or the exact calculation of areas, were intensively studied from ancient times through the period just before the invention of calculus in the 17th century. The problem of squaring the circle, arguably the most celebrated unsolved problem of antiquity, is a notable example. To ancient Greek mathematicians, including Euclid, Hippocrates, Archimedes and others, calculation of area meant ruler and compass construction of a square whose area is provably equal to that of a given figure. Thus, to square the circle, one had to construct the side of a square having area equal to that of a circle of unit radius; equivalently, to construct a length equal to $\sqrt{\pi}$ from a unit length using ruler and compass only, a construction we now know to be impossible.

Ancient Greek mathematicians could readily square polygons but had only very scattered and limited success with figures having one or more curvi-linear sides. In one of the great *tours de force* of ancient mathematics, Archimedes was able sometime before 212 BCE to square segments of parabolas cut off by chords. His results show, in particular, that if a parabola has a vertex at one corner of a rectangle, and passes through the opposite corner, then the smaller area bounded by the parablola and the rectangle is equal to 1/3 of the rectangle's area. We might say that Archimedes had successfuly calculated the integral

$$\int_0^1 x^2\,dx = \frac{1}{3}$$

using purely geometric methods.

Lines and parabolas are special cases of *power curves* having Cartesian equations of the form $y = x^k$. Thus, a natural extension of the problem solved by Archimedes is to effect the quadrature

$$\int_0^{x_0} x^k\,dx = \frac{x_0 y_0}{k+1}, \quad (y_0 = x_0^k), \tag{1.1}$$

showing that when the parabola in Archimedes' rectangle is replaced by the power curve with exponent k the appropriate fraction of the rectangle's area is $\frac{1}{k+1}$.

Following Archimedes' success with the case $k = 2$, there was essentially no progress at all on any of the higher exponents until the 17th century, when suddenly the quadrature problem was solved in full generality, and nearly simultaneously, by several mathematicians working more or less independently. It happens surpisingly often in the history of Mathematics that, progress on some problem having stalled completely, some new idea or way of thinking that leads to a breakthrough is found to be "blowing in the wind" by several people at more or less the same time.

In the case of the quadrature of power curves, the key players in the solution were Bonaventura Cavalieri (1598-1647), Pierre de Fermat (1601-1665), Gilles Persone de Roberval (1602-1675), and Blaise Pascal (1623-1662). Cavalieri conjectured the correct solution without providing a rigorous proof in the modern sense. Fermat and Roberval claimed to have rigorous proofs, and there is evidence from letters they exchanged that their claims were correct. Slightly later, Pascal outlined an argument that, though rather vague, could be made rigorous. See, for example, [1], pp. 104-109.

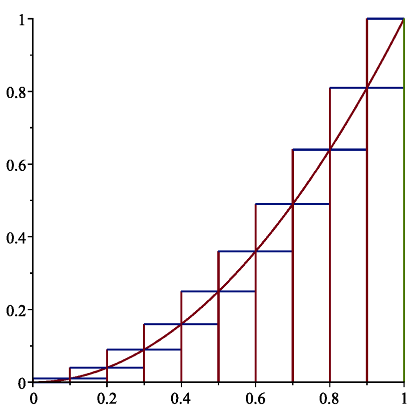
Both Fermat and Roberval based their quadratures of the power curves on the inequalities

$$\left(1+2^k+3^k+\cdots+(N-1)^k\right) < \frac{N^{k+1}}{k+1} < \left(1+2^k+3^k+\cdots+N^k\right), \tag{1.2}$$

valid for all natural numbers N and k, which they appear to have discovered independently. As we show in the following section, these inequalities readily solve the quadrature problem in conjunction with "Riemann sum" arguments that are familiar to students. Modern texts normally present only the first few cases ($k = 1, 2$) based on exact formulas for sums of the kth powers of the first n natural numbers. These formulas, though beautiful in their own right, grow more complicated with increasing k. The Fermat-Roberval inequalities, on the other hand, handle all cases with equal ease, and it is somewhat unfortunate that they seem to have disappeared from modern calculus texts.

2 Application of the Inequalities

In this section we show how to use the Fermat-Roverval inequalities to complete the quadratures of the power functions $y = x^k$ with positive integer values of k.



Before the advent of modern notation it was common for mathematicians to present arguments in the form of a "generalizable example." Such arguments depend on the choice of a particular number or other quantity, but it is clear from the argument that the quantity could have been chosen arbitrarily. The figure to the left shows two approximations to the area A under the graph of $y = x^k$ between $x = 0$ and $x = 1$. The unit interval is divided into 10 equal pieces with each piece forming the base of two rectanges, one inscribed under the curve and the other circumscribed above it. Thus, the sum of the areas of the 10 inscribed rectangles underestimates the true value of A and the sum of the areas of the circumscribed rectangles gives an overestimate. Since the first inscribed rectangle has area zero, the sum of the inscribed areas is $\frac{1}{10^{k+1}}(1+2^k+3^k+\cdots+9^k)$ and the sum of the circumscribed areas is $\frac{1}{10^{k+1}}(1+2^k+3^k+\cdots+10^k)$. It is clear that the number 10 in this example could be replaced by any natural number N, and therefore we have for each N the two-sided inequality

$$\frac{1}{N^{k+1}}(1+2^k+3^k+\cdots+(N-1)^k) < A < \frac{1}{N^{k+1}}(1+2^k+3^k+\cdots+N^k).$$

On the other hand, the Fermat-Roberval inequalities yield

$$\frac{1}{N^{k+1}}(1+2^k+3^k+\cdots+(N-1)^k) < \frac{1}{k+1} < \frac{1}{N^{k+1}}(1+2^k+3^k+\cdots+N^k).$$

The numbers on the extreme sides of these two inequalities differ only in the inclusion of one extra term of size $\frac{1}{N}$ on the right-hand side. Thus both numbers A and $\frac{1}{k+1}$ belong to the same open interval of length $\frac{1}{N}$, and therefore we have $|A - \frac{1}{k+1}| < \frac{1}{N}$. Since N can be taken arbitrarily large, we conclude that $A = \frac{1}{k+1}$. (To obtain the more general result of (1.1), rescale the units of the x-axis so that the point currently designated 1 has coordinate x_0, and the units of the y-axis so that 1 becomes $y_0 = x_0^k$.)

3 Proofs of the Fermat-Roberval Inequalities

We show that the Fermat-Roberval inequalities follow readily from the Mean Value Theorem, and that they can also be proved by induction. A proof of the Fermat-Roberval inequalities, followed by their use in deriving quadratures, as shown in the previous section, would be quite suitable for an introductory calculus course for prospective mathematics majors.

For a given natural number k, apply the Mean Value Theorem to the function $f(x) = x^{k+1}$ on the interval $[i, i+1]$, where i is also a natural number. We have that

$$(i+1)^{k+1} - i^{k+1} = (k+1)x^k,$$

for some $i < x < i+1$. Considering the extreme cases for the value of x, we have

$$(k+1)i^k < (i+1)^{k+1} - i^{k+1} < (k+1)(i+1)^k, i = 0, 1, \ldots$$

Since the sum on i of the middle members telescopes, we obtain by summing the right-hand inequalities that

$$N^{k+1} = \sum_{i=0}^{N-1}((i+1)^{k+1} - i^{k+1}) < (k+1)\sum_{i=1}^{N} i^k.$$

Similarly, by summing the left-hand sides we obtain

$$(k+1)\sum_{i=0}^{N-1} i^k < \sum_{i=0}^{N-1}((i+1)^{k+1} - i^{k+1}) = N^{k+1}.$$

Together, these comprise the Fermat-Roberval inequalities.

This argument would not have been accessible in Fermat and Roberval's time since the Mean Value Theorem was not rigorously formulated until the 19th century. (In his 1823 treatise *Resumé des leçons donées á l'Ecole Polytechnique sur le calcul infinitésimal,* Augustin-Louis Cauchy presented for the first time a form of the Mean Value Theorem sufficient for the argument above. A convenient reference for the statement is pp. 775-6 of [2].)

While Fermat did not have access to the Mean Value Theorem, he may have known about proof by induction. His contemporary, Blaise Pascal, was one of the first to use proof by induction in close to its modern form. See, e.g., [2], pp. 491-493. The inequalities

$$i^k \quad < \quad \frac{(i+1)^{k+1} - i^{k+1}}{k+1} \quad < (i+1)^k$$

yield readily to induction on k. For $k = 1$ the inequalities reduce to $i < \frac{1}{2}((i+1)^2 - i^2) = i + \frac{1}{2} < i+1$. Assuming they hold for a given $k \geq 1$ we have

$$(i+1)^{k+2} - i^{k+2} = (i+1)(i+1)^{k+1} - i^{k+2} = (i+1)\left((i+1)^{k+1} - i^{k+1}\right) + i^{k+1}.$$

By the right-hand inequality of the inductive hypothesis, the last expression is strictly less than

$$(i+1)(k+1)(i+1)^k + i^{k+1} < (k+1)(i+1)^{k+1} + (i+1)^{k+1} = (k+2)(i+1)^{k+1}.$$

Using instead the left-hand inequality of the inductive hypothesis, we obtain

$$(i+1)(k+1)i^k + i^{k+1} > (k+1)i^{k+1} + i^{k+1} = (k+2)i^{k+1}$$

as a strict lower bound, from which the desired result follows by induction. The Fermat-Roberval inequalities follow by the same telescoping sum argument given above.

Since Fermat gave no indication of being aware of this proof, one that was well within his powers to produce, we may speculate that the margins of whatever books he was reading were too small to contain it, or that perhaps he considered the argument too trivial to deserve specific mention. In the case of Roberval, there is evidence (see, e.g., [3] pp. 372-3) that he made a practise of keeping the details of his discoveries secret in order to protect his position at the Royal College in Paris. At intervals specified in his contract, he was obligated to set an examination for would-be challengers. Not surprisingly, no challenger was successful.

References

[1] C.H. Edwards,Jr. *The Historical Development of the Calculus*, Springer-Verlag, New York, 1979.

[2] V.J. Katz, *A History of Mathematics, An Introduction, 3rd Edition*, Addison Wesley, Boston, 2009.

[3] J. Suzuki , *A History of Mathematics*, Prentice Hall, Upper Saddle River, New Jersey, 2002.