Extensions of the Riemann Integral

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Abstract

We consider a family of extensions of the Riemann integral and connections with other known generalizations.

1 Introduction

The Riemann integral has a number of attractive features. Its definition as limit of Riemann sums, e.g., is itself a template for applications of the integral (distance, area, work, $\ldots$,) and it is general enough for practical purposes. Mathematicians aren't happy with its limited domain and behavior relative to pointwise limits. For example, simple functions like $\chi_{\mathbb{Q}}$, the characteristic function of the rationals, are not Riemann integrable, while being bounded pointwise limits of functions that are Riemann integrable.

Over the last century and a half, generalized Riemann integrals have been constructed to address these shortcomings by many, including Lebesgue, Denjoy, Perrone, Khintchine, Kurzweil, Henstock, and others. (See [4] for a good survey, including many of these generalizations developed in detail.)

In this paper we will explore a family of extensions obtained by adopting a slightly different point of view, one that is illustrated by the following pair of statements: (a) f is continuous almost everywhere; (b) f is equal to a continuous function almost everywhere. These say quite different things despite their apparent similarity. For example, $f = \chi_{\mathbb{Q}}$ is continuous nowhere, but equal to the zero function (continuous) almost everywhere.

We shall consider functions f that equal a Riemann integrable function g on sets that are "large" by various measures. The only integrals that actually get done are $\int_a^b g(x)\,dx$, i.e., Riemann integrals. We don't introduce any new limit processes; rather, we immediately extend the domain of Riemann integration. In effect, we are allowing a given integrand f to be "repaired" so as to become integrable, by redefining it on certain sets (i.e., redefined to be g wherever it differs.) Accordingly, we call our results extensions rather than generalizations of the Riemann integral.

The Riemann integral has a list of basic properties that one would insist any extended or generalized integral share with it. These are (here we assume f and g are unspecified members of $\mathcal{R} = \mathcal{R}([a, b])$, the set of Riemann integrable functions on $[a, b]$, throughout:)

(1.1) (Monotonicity property) $\quad f \geq g \implies \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx,$

(1.2) (Linearity property) $\quad \alpha, \beta \in \mathbb{R} \implies \int_a^b \alpha f(x) + \beta g(x)\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx,$

(1.3) (Continuity property) $\quad x \to \int_a^x f(t)\,dt$ is a continuous function of x.

For a given f the integral is a finitely additive measure as a function of interval of integration; in particular, if $a < b < c$ then

(1.4) (Additivity property) $\quad \int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$

The set $\mathcal{R}([a,b])$ is closed under uniform convergence. If $f_n \in \mathcal{R}([a,b])$ and $f_n \to f$ uniformly on $[a,b]$, then $f \in \mathcal{R}([a,b])$, and

$$(1.5)\ \text{(Completeness property)} \qquad \int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx.$$

The product fg is also Riemann integrable. This *closure under multiplication property* is useful and desirable, but does not hold for some generalizations. Both the Lebesgue and Kurzweil-Henstock integrals fail to be closed under multiplication.

The next section gives our extension procedure and 4 examples of extensions obtained thereby. Section 3 provides a version of the completeness property (1.5) valid for any of our extensions. This is discussed separately since it is more difficult to derive than the other properties.

It is desirable to have a version of the Fundamental Theorem of Calculus, and section 4 provides one for some of the extensions. The last section discusses ideas for further extensions and related matters.

2 Repaired Riemann Integrals

Lemma 2.1. *If two Riemann integrable functions agree on a dense set D, their Riemann integrals are equal.*

The proof is almost immediate from the fact that tags for partitions, no matter how fine, can always be chosen from the dense set. If so, the resulting Riemann sums are equal for the two functions.

Recall that a family $\mathcal{C}$ of subsets of $I = [a,b]$ is called a *filter of sets* if it has the following properties:

(2.1) $I \in \mathcal{C}, \emptyset \notin \mathcal{C}$

(2.2) $A \subseteq B, A \in \mathcal{C} \implies B \in \mathcal{C}$

(2.3) $A, B \in \mathcal{C} \implies A \cap B \in \mathcal{C}$

For example, if D is a designated non-empty set, the class of all subsets of I containing D is a filter called the *principal filter generated by D.* We are particularly interested in filters, all of whose members are dense subsets of I, which we shall call *dense filters.* Principal filters generated by a dense set, for example the set of rational numbers in I, are dense filters.

Definition 2.1. *Let $\mathcal{C}$ be a dense filter of subsets of $[a,b]$. A function $f \in \mathcal{CR}$ (or f is $\mathcal{C}$ integrable) if there is a Riemann integrable function g and a set $A \in \mathcal{C}$ such that $f = g$ on the set A.*

The function g is not uniquely determined by f, but the integral of g is uniquely determined: If there are two such functions $g_1 = f$ on $A \in \mathcal{C}$, and $g_2 = f$ on $B \in \mathcal{C}$, then $g_1 = g_2$ on $A \cap B$, which is also a set in $\mathcal{C}$, and hence dense. The desired conclusion is immediate from Lemma 2.1.

We may then unambiguously extend the Riemann integral from $\mathcal{R}$ to $\mathcal{CR}$ by defining

$$\int_a^b f(x)\,dx = \int_a^b g(x)\,dx.$$

The argument following Definition 2.1 is nearly a template for deducing properties (1.1)-(1.5) and closure under multiplication for functions in $\mathcal{CR}$ from their counterparts for the ordinary Riemann integral. For example, suppose $f \geq g$ and $f = g_1$ on A and $g = g_2$ on B, where g_1 and g_2 are Riemann integrable and A and B belong to $\mathcal{C}$. Then also $f = g_1 \vee g_2$ on $A \cap B$ and $g = g_1 \wedge g_2$ on $A \cap B$. But $g_1 \vee g_2 \geq g_1 \wedge g_2$ are Riemann integrable, and $\int_a^b (g_1 \vee g_2)(x)\,dx \geq \int_a^b (g_1 \wedge g_2)(x)\,dx$. The desired result then follows, since the set $A \cap B$ belongs to $\mathcal{C}$.

Linearity and closure under multiplication can be proved the same way. More care is needed for statements (1.3) and (1.4) since different intervals of integration are involved, and the choice of filter is tied to the interval of integration. Thus, in any application of these statements, we shall assume that all intervals involved are subsets of one single interval, say $[a,c]$, and the chosen filter is a filter of subsets of this interval. If f is $\mathcal{C}$ integrable on $[a,c]$, then $f\chi_{[a,b]}$ is also $\mathcal{C}$ integrable since $\chi_{[a,b]}$ is Riemann integrable. Obvious analogues of (1.3) and (1.4) then hold provided we understand that $\int_a^b f(x)\,dx$ is shorthand for $\int_a^c (f\chi_{[a,b]})(x)\,dx$.

The formulation and proof of a suitable version of the completeness property is deferred to a later section.

The extended Riemann integral obtained by using all but a few trivial filters is a strict extension of the Riemann integral. Typically, $\mathcal{C}$ will contain non-measurable sets, so any function in $\mathcal{CR}$ can be redefined on a certain non-measurable set, and the result still belongs to $\mathcal{CR}$ with the same integral. This point is addressed further in the examples that follow.

Example 2.1. *Take $\mathcal{C}$ to be the principal filter generated by the rational numbers in $[a,b]$, denoted by Q.*

A function f belongs to $Q\mathcal{R}$ if and only if it is equal to some Riemann integrable function h on the rationals, $f(x) = h(x)$, for any rational x in $[a,b]$. Any such function shares its integral with many other functions since f can be altered at will on irrational values of x. Thus, for example, the Dirichlet function $\chi_{\mathbb{Q}}$ is Q integrable because it equals the constant function 1 on the rationals.

One could consider equally well here any fixed countable dense set in place of the rationals, but the Q integrable functions are quite natural. One has, for example, the result that

$$(2.4)\quad \int_0^1 f(x)\,dx \quad = \quad \lim_{n\to\infty} \frac{2}{n^2} \sum_{i=1}^{n} \sum_{j=0}^{i} f(j/i)$$

holds for any Q integrable function f. This can be checked by direct computation when f is a polynomial. One may then deduce the same result for continuous functions from the Weierstrass approximation theorem. The same result also holds for f which are only assumed to be Riemann integrable. To see this, divide and multiply by i between the summation signs, and note that the result has the form $\frac{2}{n^2}\sum_{i=1}^{n} iR_i$, where R_i are Riemann sums converging to $\int_0^1 f(x)\,dx$ as $i \to \infty$. The result then follows by an easy summability argument.

Example 2.2. *Take $\mathcal{C}$ to be the co-countable filter, i.e., the filter consisting of subsets of $[a,b]$ whose complements are countable, denoted by CC.*

CC is dense, and the resulting integration theory is nearly identical with the Riemann theory, with the added convenience that integrable functions can be arbitrarily modified on a countable set of points without destroying integrability or changing the value of the integral. We should also mention here that the CC integral of $\chi_{\mathbb{Q}}$ is 0, while the Q integral is $b-a$. Thus, the value of an extended integral may depend on the chosen filter. (See further discussion below in section 5.)

Example 2.3. *Take $\mathcal{C}$ to be the collection of measurable sets whose complements have measure 0, denoted by Z*

Z is a dense filter since the union of sets of measure zero has measure zero. The resulting repaired Riemann integration is just Lebesgue integration restricted to Riemann integrable functions.

Example 2.4. *Take $\mathcal{C}$ to be the collection of sets that contain dense G_δ subsets, denoted by N.*

This example was introduced in [5]. Let F be a Cantor-like subset of $[0,1]$ with positive measure. (See, e.g., [3], pp 70-72.) Then F^c belongs to N, so any N integrable function equal to a Riemann integrable function on F^c can be altered arbitrarily on F. Sets of positive measure contain non-measurable sets, and the characteristic function of any such non-measurable set is therefore N integrable with integral equal to 0.

3 A Limit Theorem

We shall formulate and prove in this section a version of the completeness property (1.5) that is sufficient for each of the examples considered in the previous section.

Theorem 3.1. *Let D be a dense subset of $[a,b]$ and f_n a sequence of functions which converge uniformly on D to some function f. Let g_n be functions that are Riemann integrable on $[a,b]$ and are such that each $f_n = g_n$ on D. Then there is a Riemann integrable function h such that $f = h$ on D and*

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b h(x)\,dx.$$

The uniform limit of Riemann integrable functions is Riemann integrable, but we don't know that the functions g_n converge uniformly except on D. Accordingly, we first use the values of the g_n on D to determine new values on D^c which are controlled by the fact that the g_n are Riemann integrable. Then we show (a) that the new functions h_n are Riemann integrable, and (b) converge uniformly on $[a,b]$ to a function h.

Riemann integrability of a bounded measurable function h is tied to the behavior of two auxiliary functions $M(h)(x)$ and $m(h)(x)$ defined as follows:

$$M(h)(x) = \lim_{r\to 0} \sup\{h(y) : |x-y| < r\}, \quad m(h)(x) = \lim_{r\to 0} \inf\{h(y) : |x-y| < r\}.$$

One then has that h is Riemann integrable if and only if $\int_a^b M(h)(x) - m(h)(x)\,dx = 0$ (Here, the integral is understood in the Lebesgue sense.) See, for example, Theorem 3.14, p. 38 of [4].

It will be convenient to introduce variants of the auxiliary functions defined by

$$M_0(h)(x) = \lim_{r\to 0} \sup\{h(y) : 0 < |x-y| < r\}, \quad M_D(h)(x) = \lim_{r\to 0} \sup\{h(y) : |x-y| < r, y \in D\},$$

together with analogous definitions with M replaced by m.

Now we extend the functions g_n restricted to D to functions h_n defined by $h_n(x) = M_D(g_n)(x)$ for $x \notin D$. Then we have for each $x \in [a,b]$:

(3.1) $M_D(g_n)(x) \le M(g_n)(x)$,

(3.2) $M(h_n)(x) \le M(g_n)(x)$,

and

(3.3) $m(h_n)(x) \ge m(g_n)(x)$

Statement (3.1) is immediate from the fact that the supremum of the set on the left side is taken over a subset of the corresponding set on the right side.

To prove (3.2), first note that $M(h_n)(x) \le h_n(x) \vee M_0(h_n)(x)$. Now for a given $r > 0$, if y belongs to the set in the definition of $M_0(h_n)(x)$, we consider separately the cases that $y \notin D$ and $y \in D$. If $y \notin D$ then

$$h_n(y) = M_D(g_n)(y) \le M(g_n)(y) = \lim_{s\to 0} \sup\{g_n(y') : |y'-y| < s\}.$$

Now if $s < r - |x-y|$ then

(3.4) $\{g_n(y') : |y'-y| < s\} \subseteq \{g_n(y) : |x-y| < r\}$.

If $y \in D$ then $h_n(y) = g_n(y)$, so the supremum of the set on the right of (3.4) is an upper bound for $\sup\{h_n(y) : 0 < |x-y| < r\}$. It follows that $M_0(h_n)(x) \le M(g_n)(x)$. On the other hand, if $x \in D$ then $h_n(x) = g_n(x) \le M(g_n)(x)$, which completes the proof of (3.2).

The proof of (3.3) is similar. Starting from $m(h_n)(x) \ge m_0(h_n)(x) \wedge h_n(x)$, one shows $m_0(h_n)(x) \ge m(g_n)(x)$ using that fact that

$$\lim_{s\to 0} \sup\{g_n(y') : |y'-y| < s, y' \in D\} \ge \lim_{s\to 0} \inf\{g_n(y') : |y'-y| < s\}$$

when $y \notin D$, and $h_n(y) = g_n(y)$ when $y \in D$. Thus $m(h_n)(x) \ge m(g_n)(x) \wedge h_n(x)$. Then if $x \in D$ we have $m(h_n)(x) \ge m(g_n)(x) \wedge g_n(x) = m(g_n)(x)$, while if $x \notin D$ then $h_n(x) = M_D(g_n)(x) \ge m(g_n)(x)$, so again $m(h_n)(x) \ge m(g_n)(x)$.

Now we can show that each h_n is Riemann integrable. We have, combining (3.2) and (3.3), that $0 \le M(h_n)(x) - m(h_n)(x) \le M(g_n)(x) - m(g_n)(x)$. Integrating, we have

$$0 \le \int_a^b M(h_n)(x) - m(h_n)(x)\,dx \le \int_a^b M(g_n)(x) - m(g_n)(x)\,dx = 0,$$

since g_n is Riemann integrable.

The h_n converge uniformly to f on D, and we define the function h by $h(x) = f(x)$ on D and $h(x) = M_D(f)(x)$ on D^c. We have finally that h_n converge uniformly to h on $[a,b]$: Given $\epsilon > 0$, choose n so that $|g_n(x) - f(x)| < \frac{\epsilon}{2}$ for all $x \in D$. For any given $x \notin D$ and $r > 0$, choose $y \in D$ such that $|x - y| < r$ and $M_D(g_n)(x) \le g_n(y) + \frac{\epsilon}{2}$. Then $M_D(g_n)(x) \le f(y) + \epsilon \le \sup\{f(y) : |x-y| < r, y \in D\} + \epsilon$. Then letting $r \to 0$, we have $M_D(g_n)(x) \le M_D(f)(x) + \epsilon$. Similarly, $M_D(f)(x) \le M_D(g_n)(x) + \epsilon$.

Since $h_n \to h$ uniformly on $[a,b]$, h is Riemann integrable and

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \lim_{n\to\infty} \int_a^b h_n(x)\,dx = \int_a^b h(x)\,dx.$$

The proof of Theorem 3.1 is complete.

4 The Fundamental Theorem of Calculus (FTC)

In this section we shall assume that f is differentiable on $[a,b]$, i.e., the derivative exists at every point. One of the most attractive features of the Henstock integral is that it provides a perfect version of the fundamental theorem of calculus: The derivative f' is automatically Henstock integrable, and one has

$$\text{(4.1)} \quad \int_a^b f'(x)\,dx = f(b) - f(a).$$

This fails for the Riemann integral itself because derivatives can be quite discontinuous. Indeed, it is possible to construct a derivative that is discontinuous almost everywhere, hence not Riemann integrable. On the other hand, if one assumes that f is such that f' is Riemann integrable, then (4.1) does hold. It is natural then to ask whether such a restricted FTC holds for our extensions of the Riemann integral? Our next result shows that the answer is yes in some cases.

Theorem 4.1. *Let $\mathcal{C}$ be a dense filter of subsets of $[a,b]$ such that the complement of each $A \in \mathcal{C}$ has measure zero. Then (4.1) holds whenever f' is $\mathcal{C}$ integrable.*

Indeed, this result follows immediately from (4.1) itself, and the fact that two Henstock integrable functions that agree almost everywhere have equal integrals. (See, e.g., Theorem 9.10 of [4].)

In general, one cannot expect to have an FTC assuming only $\mathcal{C}$ integrability of the derivative function. For example, it is possible [2] to have a non-constant function f whose derivative vanishes at a dense set of points, D. For any set A in the principal filter generated by D, we have that f is equal on A to a Riemann integrable function, i.e., the zero function. More elaborate examples show that there is no reasonable FTC for example 2.4. It is possible, for example, to construct a non-constant differentiable function f whose derivative vanishes on a dense open set. See [1].

The other side of the FTC, including characterization of which functions are indefinite integrals, is not of interest since it is the same as for the Riemann integral. (A function is an indefinite Riemann integral if and only if it is absolutely continuous with a derivative that is continuous almost everywhere.)

5 Unbounded Functions

Since Riemann integrable functions are bounded, our extensions are limited to functions that are bounded, or at least bounded on some set belonging to the filter. In this section we briefly sketch a further extension procedure that may be applied to some unbounded functions. Let $A_n, n = 1, 2, \dots,$ be sets that partition the given interval $[a, b]$, i.e., the A_n are pairwise disjoint with union equal to $[a, b]$. We say that this partition is *compatible with* a given dense filter $\mathcal{C}$ if, for any $f \in \mathcal{CR}$, we have $f\chi_{A_n} \in \mathcal{CR}$ for each n, and

$$\int_a^b f(x)\,dx \;=\; \sum_{n=1}^{\infty} \int_a^b f(x)\chi_{A_n}(x)\,dx. \tag{5.1}$$

Here all integrals are understood in the $\mathcal{CR}$ sense.

Compatibility is automatic, regardless the dense filter, when the A_n are intervals, or, more generally, when the χ_{A_n} are each Riemann integrable. To see this, let $g \in \mathcal{R}$ and $B \in \mathcal{C}$ be such that $f = g$ on B. Then we also have

$$g(x) \;=\; \sum_{n=1}^{\infty} g(x)\chi_{A_n}(x),\;\; x \in B,$$

and

$$\int_a^b f(x)\,dx \;=\; \int_a^b g(x)\,dx \;=\; \sum_{n=1}^{\infty} \int_a^b g(x)\chi_{A_n}(x)\,dx,$$

by the Bounded Convergence Theorem. But $g\chi_{A_n} \in \mathcal{R}$ for each n, $f\chi_{A_n} = g\chi_{A_n}$ on B, and (5.1) follows.

For a non-negative function f, we say $f \in \mathcal{C}^*\mathcal{R}$ (or f is $\mathcal{C}$ integrable in the extended sense) if there is a partition compatible with $\mathcal{C}$ for which $f\chi_{A_n} \in \mathcal{CR}$ for each n and the series on the right-hand side of (5.1) converges. Then we use the value of the sum as the definition of the left-hand side of (5.1). To see that the resulting extended integral is well-defined, let B_n be some other partition compatible with $\mathcal{C}$. Note that for each m we have

$$\int_a^b f(x)\chi_{B_m}(x)\,dx \;=\; \sum_{n=1}^{\infty} \int_a^b f(x)\chi_{B_m}(x)\chi_{A_n}(x)\,dx,$$

by (5.1). The desired result then follows easily, since

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \int_a^b f(x)\chi_{B_m}(x)\chi_{A_n}(x)\,dx \;=\; \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \int_a^b f(x)\chi_{B_m}(x)\chi_{A_n}(x)\,dx.$$

It is easy to check that this extended-sense integral is monotone and linear. Following the standard pattern, we declare $f \in \mathcal{C}^*\mathcal{R}$ when f is not assumed non-negative if and only if $|f| \in \mathcal{C}^*\mathcal{R}$, and define

$$\int_a^b f(x)\,dx \;=\; \int_a^b f_+(x)\,dx - \int_a^b f_-(x)\,dx.$$

Again, it is straightforward to check that the resulting $\mathcal{C}^*$ integral is monotone and linear.

Example 5.1. $x^{-\alpha} \in \mathcal{C}^*\mathcal{R}$ *on* $(0, 1]$ *when* $\alpha < 1$.

Here we may use $A_n = (a_{n+1}, a_n]$, where $a_1 = 1$ and $a_n \downarrow 0$ as $n \to \infty$.

6 Discussion

In a bid for the title of most general integral, two approaches to further generalize the Riemann integral suggest themselves. First, one might hope to find filters that are "optimal" in some sense. In general, larger filters containing smaller sets are better. Could there be some filter for which our integration extends the Henstock theory? Unfortunately this is clearly not possible: Our repaired Riemann integrals are always closed under products, while Henstock integration is not. It is also not useful to consider ultrafilters, since there are no dense ultrafilters.

A second possibility is to replace the Riemann integral in its role as integral to be repaired. The Lebesgue integral does not appear to be suitable since it already integrates equivalence classes of functions. The

Henstock integral, in any of its equivalent forms, also runs into the immediate problem that Lemma 2.1 fails. (For example, the zero function and the characteristic function of the irrationals agree on the set of rationals, both are Henstock integrable, and their integrals are 0 and 1 respectively.) One way around this problem would be to consider *determining* filters rather than dense filters. A set A is *determining* if, whenever two Henstock integrable functions agree on A, their (Henstock) integrals are equal. A filter of determining sets would then be called a determining filter.

The Z filter (see example 2.3) is a determining filter. In this case, however, the resulting repaired Henstock integration is identical with the Henstock theory (See Theorem 9.10 of [4].) The N filter of example 2.4 is not determining, and it is not clear if there are any interesting examples.

The following example raises a legitimate philosophical objection to extension of the Riemann integral by the methods of section 3.

Example 6.1. *Let f be bounded and measurable on $[0,1]$, but not Riemann integrable. Then $M(f) > m(f)$ on a set of positive measure, where $M(f)$ and $m(f)$ are the functions defined in section 3. Suppose, in addition, that these functions are each Riemann integrable, and that $f = M(f)$ on some dense subset of $[0,1]$, and $f = m(f)$ on some dense subset also. (These conditions hold, e.g., if $f = \chi_{\mathbb{Q}}$.) Then, for any real number α that satisfies*

$$\int_0^1 m(f)(x)\,dx \quad < \quad \alpha \quad < \quad \int_0^1 M(f)(x)\,dx,$$

there is a dense filter $\mathcal{C}$ such that $f \in \mathcal{CR}$ with extended integral equal to α.

To see this, first use the Intermediate Value Theorem to find a number $c \in [0,1]$ such that

$$\int_0^c m(f)(x)\,dx + \int_c^1 M(f)(x)\,dx = \alpha.$$

Let D be a dense subset of $[0,1]$ such that $f = m(f)$ on $D \cap [0,c]$ and $f = M(f)$ on $D \cap (c,1]$. Let $\mathcal{C}$ be the principal filter generated by D and $g = m(f)\chi_{[0,c]} + M(f)\chi_{(c,1]}$. Then g is Riemann integrable, $f = g$ on a set in $\mathcal{C}$, and $\int_0^1 g(x)\,dx = \alpha$.

It would be of interest to determine whether the result of Example 6.1 holds for any bounded measurable function.

Examples such as these raise the question: if several extensions of the Riemann integral assign different values to the integral of a function that is not Riemann integrable, which value is the “right” one? In a more philosophical vein, what do we mean by “right”?

Consider the following thought experiment. Suppose we move an object from point a to point b, pushing against a variable force given by $f(x)$. According to the physical definition of work, we do an amount of work equal to the (Riemann) integral from a to b of f. We can imagine a force, provided perhaps by some demon pushing back against us, that is not Riemann integrable. In such a case, we might simply define the work done to be the value given by the Lebesgue integral (assuming f is Lebesgue integrable), but is that the proper measure of the fatigue we would experience?

Of course, our experiment merely calls into question the use of abstractions like the Lebesgue integral outside of mathematics. On the other hand, we can frame similar objections in a purely mathematical context. For example, suppose f is an L^2 function that is not Riemann integrable. We can use the Lebesgue integral to calculate Fourier coefficients for f, and the resulting Fourier series is known to converge to f almost everywhere. Using some of our extensions, however, would give “wrong” answers for these coefficients, because the resulting series would either fail to converge, or would converge to the wrong function.

References

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