Repairing the Discontinuous Function

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Abstract

Some discontinuities can be removed by redefining a function on a suitably small set. We also consider restoring Riemann integrability by redefining the integrands.

1 Introduction

Many functions are very badly behaved. The characteristic function of the rationals, $\chi_{\mathbb{Q}}$, for example, is discontinuous at every point, and even has a discontinuity of the second kind at every point. Faced with this reality, one often tries to "repair" a defective function in ways that are compatible with an intended use. Even in calculus classes, it is very convenient to allow functions to be redefined at isolated points, e.g., in order to fill in obvious holes in their graphs.

If we allow redefinition of functions on other types of small sets then even functions like $\chi_{\mathbb{Q}}$ can be tamed. In probability theory, for example, it is useful to allow probability density functions to be redefined on sets of Lebesgue measure zero, since this has no effect on their use in calculating probabilities and may improve the functions in other ways. In Lebesgue integration a function can be altered arbitrarily on a set of measure zero without harming the integrability or changing the value of the integral.

Can continuity properties of a function be improved by changing the function on small sets? The function $\chi_{\mathbb{Q}}$ can be made continuous - constant even - by redefining its values on a countable set. For general measurable functions, though, it may not be possible to introduce continuity at even a single additional point by redefining the function on a set of measure zero. (See the example at the end of section 2.) If we allow a function to be redefined on a set of first category, i.e., one that is topologically small, then the situation is quite different. We show in section 2 that any Borel function can be redefined on a set of first category so as to become continuous at a dense set of points. This result, which deserves to be better known, appears to go back to the work [1] of Blumberg in 1922. It may be that our proof has some independent interest or is more accessible than the original.

A word of caution is in order about the phrase "f can be made continuous on a subset E", as it can plausibly have one of at least 3 distinct meanings: (i) the restriction of f to E is continuous on E; (ii) there is a continuous function that equals f at each point of E; (iii) the set E is a subset of the set of continuity points of f. The last interpretation, arguably the strongest, is the one we intend in the present paper. Well-known results in real analysis such as Lusin's Theorem (see, e.g., [2], Theorem 8.2,) on the other hand, deal with the first interpretation. Woe unto the PhD oral examinee who fails to fully understand the distinction!

In the final section we explore the idea of allowing redefinition on *everything but* a dense set in order to obtain an extension of the Riemann integral.

2 Continuity of Baire Measurable Functions

Let X be a Baire space, for example, a complete metric space or a locally compact Hausdorff space. (A topological space is a Baire space if the intersection of countably many dense open sets is dense.) The class of sets having the property of Baire is the smallest sigma algebra containing the open sets and the sets of first category. A real-valued function measurable with respect to this sigma algebra is said to be Baire measurable.

Theorem 2.1. *If $f : X \to \mathbf{R}$ is Baire measurable, then f can be modified on a set of first category so as to be continuous on a dense subset of X.*

First suppose f satisfies $0 < f < 1$ and introduce measurable subsets $A_j = \{t :$ the jth binary digit of $f(t) = 1\}$. Then we have

$$f(t) = \sum_{j=1}^{\infty} 2^{-j} \chi_{A_j}(t).$$

Write $A_j = F_j \Delta Q_j$, where F_j is closed and Q_j has first category. See Theorem 4.1 of [2]. Let S be the set consisting of points that belong to any Q_j or to the boundary of any of the F_j. The set S is also of first category, since the boundary of a closed set is closed and nowhere dense.

Let

$$\tilde{f}(t) = \sum_{j=1}^{\infty} 2^{-j} \chi_{F_j^o}(t),$$

where F_j^o denotes the interior of F_j. Then $\tilde{f}$ agrees with f on the complement of S and is continuous on $G = \bigcap_{j=1}^{\infty} (\partial F_j)^c$. The latter follows since each

$$\tilde{f}_N = \sum_{j=1}^{N} 2^{-j} \chi_{F_j^o}(t)$$

is continuous on G and we have $\tilde{f}_N \to \tilde{f}$ uniformly. We are done since G is a dense G_δ.

In the general case, apply the argument just given to $\Phi(f)$, where $\Phi : \mathbf{R} \to (0, 1)$ is a homeomorphism.

In the special case $X = \mathbb{R}$ the Baire measurable functions coincide with the Borel functions, so as a corollary we obtain that any Borel measurable function can be redefined on a set of first category so as to be continuous at all points belonging to a dense G_δ set. Such a set can have Lebesgue measure zero, so one might wonder whether it is possible to redefine a Borel function on a set of first category so that it is continuous at almost all points? (Such a function, if bounded, would be Riemann integrable on any bounded interval.) The answer, unfortunately, is no in general, as we show below.

The theorem has no reasonable analogue for Lebesgue measurable functions as they are normally defined. Recall that a Lebesgue measurable function is really an equivalence class of functions whose members differ on sets of Lebesgue measure zero. Selecting a representative amounts to an arbitrary redefinition on a set of measure zero. Allowing a subsequent redefinition on a set of first category effectively allows any representative of the equivalence class to be redefined on an arbitrary set, since any subset of the real line can be written as the union of set of first category and a set of measure zero. See Corollary 1.7 of [2].

We proceed to construct an example of a bounded Borel function on [0,1] that cannot be modified on any set of first category so as to become continuous almost everywhere. As a warmup, consider the function $f(x) = \sin(1/x)$. Can we modify this function on a set of first category in such a way as to remove the oscillatory discontinuity at 0? It is not immediately obvious that the answer is no since a set of first category can be quite a large set in terms of measure. Indeed, there are first category subsets of [0,1] that have Lebesgue measure equal to 1. Nevertheless, it is not possible, and to see this we can argue as follows: take a sequence of nonempty open subintervals I_j tending to 0 on which f has alternately values greater than 1/2 and less than -1/2. For any set Q of first category $I_j \setminus Q$ is non-empty for all j, so no matter how f is modified on Q the resulting function will still fail to be continuous at zero.

Now let F be a Cantor-like subset of [0,1] having positive measure. See, for example, [3], pp.70-72. Define f to have the value zero on F and otherwise be given by

$$f(x) = \sin\left(\frac{1}{d(x, F)}\right), \ x \notin F,$$

where $d(x, F)$ denotes the distance from x to F. The complement of F consists of countably many nonempty open intervals and any neighborhood of a point of F contains infinitely many of these intervals. In turn, each such interval contains non-empty open intervals accumulating towards an endpoint on which f is alternately greater than 1/2 and less than -1/2. Thus f is not continuous at any point of F and this cannot be repaired by modifying f on any set of first category. We remark that the function f is of the first Baire class, since it is the pointwise limit of the continuous functions f_n given by

$$f_n(x) = \sin\left(\frac{1}{d(x,F) \vee \frac{1}{2\pi n}}\right).$$

3 Can the Riemann Integral be Repaired?

Everybody knows that the Riemann integral has certain defects that the Lebesgue integral "repairs": there are functions it ought to integrate that it does not; and it is not closed under bounded point-wise limits, in the sense that such a limit of eminently Riemann integrable functions (continuous functions, for example,) may fail to be Riemann integrable. Nevertheless, it suffices for most practical purposes, and is by far the more widely known; and arguably, the more widely understood.

If we set out to develop an extension of the Riemann theory that addresses some of its shortcomings, while falling short of the full sophistication of the Lebesgue theory, what form might it take? Optimistically, we might hope to integrate all bounded Borel functions, so we seek an operator T that maps bounded Borel functions to Riemann integrable functions. The function $T(f)$ represents the "repaired" version of f. Our repaired Riemann integral of f is then actually the integral of $T(f)$:

$$\int_a^b f(x)\,dx = \int_a^b T(f)(x)\,dx,$$

where the integral on the right is understood in the Riemann sense, and serves as definition of the integral on the left.

If the only goal is to produce Riemann integrable functions then this is readily accomplished, but the exercise has little value unless the function $T(f)$ is closely related to f. To maximize chances of success we shall adopt what seems to be the weakest reasonable form of association between $T(f)$ and f: Let us say that one function is a *version* of another if and only if the two functions agree on a dense G_δ set. Thus, we require of T that

$$T(f) \text{ is a version of } f. \tag{3.1}$$

(The restriction to G_δ sets is necessary in order for the relation of being versions to be transitive, and hence an equivalence relation. This is so because the intersection of two dense G_δ sets is a dense G_δ by the Baire Category Theorem.)

It does not seem unreasonable to request that T be linear, or, more precisely, that $\alpha T(f) + \beta T(g)$ be a version[1] of $T(\alpha f + \beta g)$. We should have $T(f) = f$ if f is already Riemann integrable, and to address the limit issue it seems natural to require

$$T(\lim_{n\to\infty} f_n) = \lim_{n\to\infty} T(f_n), \tag{3.2}$$

where the limit is taken in the sense of point-wise convergence.

Unfortunately the example f at the end of the last section shows that it is impossible to construct any such T. A function that agrees with f on a dense set of points must fail to be continuous at every point of F, and so it is not Riemann integrable. This is incompatible with (3.1). We must either restrict the domain of T or require a stronger form of convergence in (3.2). The first is ineffective without the latter because the

[1]This gets tedious, so for the rest of this paragraph we understand the = relation to mean "is a version of"

function f is of the first Baire class, and all of these are available as limits as soon as we include continuous functions.

As is well known, there is already a stronger form of convergence that avails - uniform convergence. Thus, any significant extension of the Riemann theory must introduce a notion of function convergence that is intermediate between uniform convergence and point-wise convergence, while at the same time increasing the domain of T strictly beyond the Riemann integrable functions. Let us be content to show here that at least some non-trivial extension is possible.

We shall say that a function f is *of Riemann class* if it has a Riemann integrable version. Such functions need not be Riemann integrable. For example, the characteristic function of a Cantor-like set of positive measure is of Riemann class, but is not Riemann integrable. It is possible to define the integral unambiguously for a function of Riemann class. Suppose f_1 and f_2 are two Riemann integrable functions that equal f on respective dense G_δ sets G_1 and G_2. Then $f_1 = f_2$ on $G = G_1 \cap G_2$, which is dense by the Baire Category Theorem. On the other hand, both f_1 and f_2 are continuous except at points of a certain set N of measure zero. But any point x not in N is a limit point of points in G, and it follows that $f_1(x) = f_2(x)$. Since these two functions are equal almost everywhere, their integrals (Riemann or Lebesgue) must coincide.

It is then easy to formulate a "convergence theorem" for the extended integral:

Theorem 3.1. *Suppose f_n are of Riemann class with Riemann integrable versions g_n that converge uniformly on $[a,b]$ to some function f. Then $f_n(x)$ converges to $f(x)$ on a dense subset of $x \in [a,b]$ and*

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 f(x)\,dx.$$

The proof is obvious.

References

[1] Henry Blumberg, New Properties of All Real Functions, *Trans. Amer. Math. Soc.,* **9**(1922), 113-129.

[2] John C. Oxtoby, *Measure and Category*, 2nd Edition, Springer-Verlag, New York, 1987.

[3] Edwin Hewitt and Karl Stromberg, *Real and Abstract Analysis*, Springer-Verlag, New York, 1975.